

Twistor lifts and factorization for conformal maps of a surface

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Abstract

A conformal map from a Riemann surface to the four-dimensional Euclidean space is explained in terms of its twistor lift. A local factorization of a differential of a conformal map is obtained. Constrained Willmore surfaces, super-conformal maps, minimal surfaces and Lagrangian surfaces are explained by twistor lifts. The factorization of a differential provides an upper bound of the area of a super-conformal map and a minimal surface around a branch point.

1 Introduction

In classical surface theory, we consider an oriented surface to be the image of an isometric immersion from a two-dimensional oriented Riemannian manifold. To investigate the Riemannian geometric properties, we frequently employ a complex structure that is compatible with the metric and orientation of a two-dimensional Riemannian manifold. We employ theory of holomorphic functions, Riemann surfaces and holomorphic vector bundles. This method is successful and has been investigated in various studies. For example, several important examples of minimal surfaces in Euclidean space are constructed by a meromorphic function and a holomorphic one-form on a Riemann surface by the Weierstrass representation formula [30], [9]. The Hopf theorem for constant mean curvature surfaces is proven by the holomorphic Hopf differential [16].

Treating a two-dimensional oriented Riemannian manifold as a Riemann surface involves treating the consideration of an isometric immersion as a conformal immersion. Therefore, we can expect that the theory of conformal immersions is closely related to the theory of isometric immersions.

A holomorphic function is a (branched) conformal immersion, and the theory of holomorphic functions is a successful theory. We obtain an idea for constructing a theory of conformal immersions, which includes the theory of holomorphic functions.

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The paper [26] seems to be an initial significant achievement using by this idea. They refer to a branched conformal immersion from a Riemann surface to the four-dimensional Euclidean space \mathbb{E}^4 as a *conformal map*. A conformal map is considered to be a holomorphic map from a Riemann surface to the four-dimensional Euclidean space with respect to an almost complex structure along f . The subsequent papers show that this approach is fruitful. For example, [10] introduces quaternionic holomorphic curves, which include holomorphic curves in complex projective space and obtains theorems that hold for holomorphic curves in complex projective space as special cases. The paper [3] explains the relation between conformal maps and spectral curves.

The almost complex structure along a conformal map is considered to be a map from a Riemann surface to the twistor space of \mathbb{E}^4 . A twistor lift is a pair that consists of a conformal map and an almost complex structure along the conformal map. An almost complex structure is invariant under conformal transforms of \mathbb{E}^4 . We can expect that the twistor space is useful for studying conformal maps. We implement this idea in this paper.

The contents of this paper are classified into two types.

The first type involves rephrasing and refining known results by twistor lifts. The twistor theory serves an important role in the study of surfaces in four-dimensional Riemannian manifolds, in particular, minimal surfaces (see [7], [11] and [4], for example). Quaternionic holomorphic geometry [5] is another useful theory for studying the surfaces in the special case. To clarify the relation between these theories, we begin recalling the twistor space of \mathbb{E}^4 after Salamon [27]. In Section 3, we define a conformal map using a map from a Riemann surface to the twistor space and explain that this definition coincides with the definition of a conformal map in [26]. Among twistor lifts of conformal maps, we distinguish a special lift that we refer to as a *canonical lift*. We express transforms of conformal maps in terms of canonical lift in Section 5. We connect a canonical lift with the Weierstrass representation in [26] in Section 6. We characterize constrained Willmore conformal maps, super-conformal maps, minimal conformal maps, and (Hamiltonian stationary) Lagrangian conformal maps in terms of canonical lifts in the subsequent sections.

The second type involves giving an upper bound of the area of a conformal map from an open disk that is centered at a branch point. In Section 3, we show that the differential of a conformal map is factored by two maps into $\mathrm{Sp}(1)$ and a $(1, 0)$ -form locally. We refer to it as a *canonical factorization*. We note that the $(1, 0)$ -form explains the intrinsic Riemannian geometry of a conformal map. The maps into $\mathrm{Sp}(1)$ give the generalized Gauss map of a surface. As a reference case, we obtain an upper bound of the area of a holomorphic function on a unit disk. We assume that f is a holomorphic function on a unit disk D with branch point at 0 of order $m - 1$ and $|f_z(z)/z^{m-2}| < C$. Applying the Schwarz lemma [28], we demonstrate in Lemma 6 that the area of f on $D_r = \{z \in \mathbb{C} : |z| < r\}$ is equal to or less than

$$\pi \frac{C^2}{m} r^{2m}.$$

If there exists $z_0 \in D \setminus \{0\}$ such that $|f_z(z_0)| = C|z_0|^{m-1}$ or $|(f_z/z^{m-2})_z(0)| = C$, then the equality holds.

The estimate for the area of a general conformal map around a branch point is more involved. We restrict ourselves to holomorphic maps, super-conformal maps and minimal conformal maps. In Section 9, we discuss minimal conformal maps. We consider a minimal conformal map of finite area from an open unit disk to the four-dimensional Euclidean space branched at the origin. A lower bound of the area is obtained in [1]. For an upper bound, we employ a variant of a canonical factorization of the differential of a minimal conformal map. By Lemma 17, two nonwhere-vanishing holomorphic map \mathbf{a} and \mathbf{b}^{-1} into \mathbb{C}^2 and a holomorphic function h vanishing at the origin are associated with the differential of a minimal conformal map $f: D \rightarrow \mathbb{R}^4$ with branch point of order $m-1$ at the origin. We assume that $|\mathbf{a}| \leq C_a$, $|\mathbf{b}^{-1}| \leq C_{b^{-1}}$ and $|h(z)/z^{m-2}| \leq C_h$ on the unit disk. In Theorem 9, we demonstrate that the area of the minimal conformal map on D_r is equal to or smaller than

$$\pi \frac{C_a^2 C_{b^{-1}}^2 C_h^2}{m} r^{2m}.$$

If $|\mathbf{a}| = C_a$, $|\mathbf{b}^{-1}| = C_{b^{-1}}$ and there exists $z_0 \in D \setminus \{0\}$ such that $|h(z_0)| = C_h|z_0|^{m-1}$ or $|(h/z^{m-2})_z(0)| = C_h$, then the equality holds. A holomorphic function is a planar minimal conformal map. We note that the upper bound for a planar minimal conformal map is reduced to the upper bound for a holomorphic function. We obtain a similar upper bound for an area of a super-conformal map in Theorem 6 in Section 8.

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2 Preliminaries

Throughout this paper, all manifolds and maps are assumed to be smooth. We review the twistor space of \mathbb{E}^4 after Salamon [27].

2.1 Elementary representation theory

Let V be a real four-dimensional vector space and let $\langle \cdot, \cdot \rangle$ be an inner product on V . We denote the norm of $v \in V$ by $|v|$. Let (J_1, J_2, J_3) with $J_1 \circ J_2 = J_3$ be a hypercomplex structure of V such that $\langle \cdot, \cdot \rangle$ is Hermitian with respect to (J_1, J_2, J_3) .

We consider V to be a right quaternionic module by

$$v(a_0 + a_1i + a_2j + a_3k) = va_0 - (J_1v)a_1 - (J_2v)a_2 - (J_3v)a_3$$

for $v \in V$ and $a_0, a_1, a_2, a_3 \in \mathbb{R}$.

Fix $v_0 \in V$ with $|v_0| = 1$. Define quaternionic linear automorphisms \tilde{J}_1, \tilde{J}_2 , and \tilde{J}_3 of V by $\tilde{J}_n v_0 = -J_n v_0$ ($n = 1, 2, 3$). Then $(\tilde{J}_1, \tilde{J}_2, \tilde{J}_3)$ is a hypercomplex

structure of V with $\tilde{J}_1 \circ \tilde{J}_2 = \tilde{J}_3$ such that $\langle \cdot, \cdot \rangle$ is Hermitian with respect to $(\tilde{J}_1, \tilde{J}_2, \tilde{J}_3)$. We consider V to be a left quaternionic module by

$$(a_0 + a_1i + a_2j + a_3k)v = a_0v + a_1\tilde{J}_1v + a_2\tilde{J}_2v + a_3\tilde{J}_3v.$$

We note that $iv_0 = v_0i$, $jv_0 = v_0j$ and $kv_0 = v_0k$. Then, V is isomorphic to the non-commutative associative algebra of all quaternions \mathbb{H} . The vector $v_0\lambda = \lambda v_0 \in V$ with $\lambda \in \mathbb{H}$ is identified with $\lambda \in \mathbb{H}$. The set $U = \{v_0\lambda : \lambda \in \mathbb{C}\}$ is identified with the set of all complex numbers \mathbb{C} .

We obtain an orthogonal decomposition of V by real vector spaces

$$V = V_c \oplus V_c^\perp, \quad V_c = \{v_0r : r \in \mathbb{R}\}.$$

Then, V_c is identified with the set $\text{Re } \mathbb{H}$ of all real parts of quaternions and V_c^\perp is identified with the set $\text{Im } \mathbb{H}$ of all imaginary parts of quaternions. We denote the quaternionic conjugate of $v \in V \cong \mathbb{H}$ by \bar{v} .

If we consider V to be a right complex vector space with the complex structure $-J_1$, then we denote it by V_+ . We obtain $V_+ = U \oplus kU$. If we consider V to be as a left complex vector space with complex structure $-\tilde{J}_1$, then we denote it by V_- . We obtain $V_- = U \oplus Uj$.

For any $v \in V$ with $|v| = 1$, a quadruplet $(v, -J_1v, -J_3v, -J_2v)$ is an orthonormal basis of V . The ordered orthonormal basis

$$\tilde{v}_0 := (v_0, -J_1v_0, -J_3v_0, -J_2v_0)$$

determines an orientation. When we identify V with \mathbb{H} , the ordered basis \tilde{v}_0 is identified with $(1, i, k, j)$. The set of all orthonormal ordered bases (v_1, v_2, v_3, v_4) with the same orientation as \tilde{v}_0 constitutes the special orthogonal group $\text{SO}(4)$ by the relation

$$(v_1, v_2, v_3, v_4) = (v_0, -J_1v_0, -J_3v_0, -J_2v_0)\beta, \quad \beta \in \text{SO}(4).$$

Let I be an orthogonal complex structure of V . The set of all orthonormal ordered bases of the form $(v_1, -Iv_1, v_2, -Iv_2)$ with the same orientation as \tilde{v}_0 constitutes a subgroup of $\text{SO}(4)$, which is isomorphic to the unitary group $\text{U}(2)$ by

$$(v_1, -Iv_1, v_2, -Iv_2) = (v_0, -J_1v_0, -J_3v_0, -J_2v_0)\beta, \quad \beta \in \text{U}(2).$$

The set of all orthonormal ordered bases of the form $(v_1, -J_1v_1, -J_3v_0, -J_2v_0)$ with the same orientation as \tilde{v}_0 constitutes a subgroup of $\text{U}(2)$, which is isomorphic to the unitary group $\text{U}(1)$ by

$$(v_1, -J_1v_1, -J_3v_0, -J_2v_0) = (v_0, -J_1v_0, -J_3v_0, -J_2v_0)\beta, \quad \beta \in \text{U}(1).$$

The group $\text{U}(1)$ is isomorphic to the set of all unit complex numbers. Similarly, the set of all orthonormal ordered bases of the form $(v_0, -J_1v_0, v_2, -J_1v_2)$ with

the same orientation as \tilde{v}_0 constitutes a subgroup of $U(2)$, which is isomorphic to the unitary group $U(1)$ by

$$(v_0, -J_1 v_0, v_2, -J_1 v_2) = (v_0, -J_1 v_0, -J_3 v_0, -J_2 v_0)\beta, \quad \beta \in U(1).$$

The set of all orthonormal ordered bases of the form $(v, -J_1 v, -J_3 v, -J_2 v)$ with the same orientation as \tilde{v}_0 constitutes a subgroup of $SO(4)$, which is isomorphic to the symplectic group $Sp(1)$ by

$$(v, -J_1 v, -J_3 v, -J_2 v) = (v_0, -J_1 v_0, -J_3 v_0, -J_2 v_0)\beta, \quad \beta \in Sp(1).$$

The symplectic group $Sp(1)$ is isomorphic to the group of all unit quaternions.

A double-covering $\phi: Sp(1) \times Sp(1) \rightarrow SO(4)$ is defined by

$$\begin{aligned} & (av_0 b^{-1}, a(-J_1 v_0)b^{-1}, a(-J_3 v_0)b^{-1}, a(-J_2 v_0)b^{-1}) \\ &= (v_0, -J_1 v_0, -J_3 v_0, -J_2 v_0)\phi(a, b), \quad (a, b) \in Sp(1) \times Sp(1). \end{aligned}$$

Because $\phi(a, a)$ preserves the decomposition $V_c \oplus V_c^\perp$ with orientation, the set of all matrices of the form $\phi(a, a)$ ($a \in Sp(1)$) constitutes the subgroup of $SO(4)$, which is isomorphic to $SO(3)$. The map ϕ composed with the inclusion $a \mapsto (a, a)$ of $Sp(1)$ into $Sp(1) \times Sp(1)$ is a double-covering $Sp(1) \rightarrow SO(3)$.

The maps $\phi|_{U(1) \times Sp(1)}$ and $\phi|_{Sp(1) \times U(1)}$ are double-coverings of $U(2)$. Selecting the double-covering $\phi|_{U(1) \times Sp(1)}$, we obtain

$$SO(4)/U(2) \cong (Sp(1) \times Sp(1))/(U(1) \times Sp(1)) = Sp(1)/U(1).$$

We fix a complex line $L = \{v_0 \lambda : \lambda \in \mathbb{C}\}$ in V_+ . Let $a = a_0 + a_1 i + a_2 k + a_3 j$ ($a_0, a_1, a_2, a_3 \in \mathbb{R}$). Then, $aL = \{(v_0(a_0 + a_1 i) - (J_3 v_0)(a_2 + a_3 i))\lambda : \lambda \in \mathbb{C}\}$ is a complex line. Let (W_0, W_1) be a holomorphic coordinate of V_+ such that $V_+ = \{v_0 W_0 - (J_3 v_0)W_1 : W_0, W_1 \in \mathbb{C}\}$ and let $[W_0, W_1]$ be the homogeneous coordinate of $\mathbb{P}(V_+)$. Then, $aL = [a_0 + a_1 i, a_2 + a_3 i]$. For $a \in Sp(1)$, we denote $aU(1) \in Sp(1)/U(1)$ by a^b . The correspondence $a^b \mapsto aL$ for any $a \in Sp(1)$ identifies $Sp(1)/U(1)$ with $\mathbb{P}(V_+)$.

Consider $Sp(1)$ as the three-dimensional sphere $S^3 = \{a \in \mathbb{H} : |a| = 1\}$. Let S^2 be the two-dimensional sphere $\{a \in \text{Im } \mathbb{H} : |a| = 1\}$. We obtain the Hopf map $H: S^3 \rightarrow S^2$, $H(a) = aia^{-1}$ of the Hopf fibration. The map $\Phi_+: Sp(1)/U(1) \rightarrow S^2$ defined by $\Phi_+(a^b) = aia^{-1}$ identifies $Sp(1)/U(1)$ with S^2 .

There is a bijective map I_+ from $Sp(1)/U(1)$ to the set of all orthogonal complex structures of V such that

$$(v_1, -I_+(a^b)v_1, v_2, -I_+(a^b)v_2) = (v_0, -J_1 v_0, -J_3 v_0, -J_2 v_0)\phi(a, b).$$

Similarly, selecting the double-covering $\phi|_{U(1) \times Sp(1)}$, we obtain

$$SO(4)/U(2) \cong (Sp(1) \times Sp(1))/(Sp(1) \times U(1)) = Sp(1)/U(1).$$

For $b^{-1} \in Sp(1)$, we denote $U(1)b^{-1} \in Sp(1)/U(1)$ by $(b^{-1})^\sharp$. The correspondence $(b^{-1})^\sharp \mapsto Lb^{-1}$ for any $b^{-1} \in Sp(1)$ identifies $Sp(1)/U(1)$ with $\mathbb{P}(V_-)$.

The map $\Phi_-: \text{Sp}(1)/\text{U}(1) \rightarrow S^2$ defined by $\Phi_-((b^{-1})^\sharp) = bib^{-1}$ identifies $\text{Sp}(1)/\text{U}(1)$ with S^2 .

The bijective map I_- from $\text{Sp}(1)/\text{U}(1)$ to the set of all orthogonal complex structures of V exists such that

$$(v_1, -I_-((b^{-1})^\sharp)v_1, v_2, -I_-((b^{-1})^\sharp)v_2) = (v_0, \tilde{J}_1 v_0, \tilde{J}_3 v_0, \tilde{J}_2 v_0)\phi(a, b).$$

Then,

$$\begin{aligned} (v_1, -Iv_1, v_2, -Iv_2) &= (v_0, \tilde{J}_1 v_0, \tilde{J}_3 v_0, \tilde{J}_2 v_0)\phi(a, b), \\ -I &= -I_+(a^\flat) = -I_-((b^{-1})^\sharp). \end{aligned}$$

We note that

$$-I_-((b^{-1})^\sharp)v = v bib^{-1}, \quad v \in V.$$

For $\beta \in \text{Sp}(1)/\text{U}(1)$ with $\beta = (b^{-1})^\sharp$, we exchange the notation $I_-(\beta)$ with \mathcal{I}_-^β :

$$-\mathcal{I}_-^\beta v = v\Phi_-(\beta).$$

For $\alpha \in \text{Sp}(1)/\text{U}(1)$, define the orthogonal complex structure \mathcal{I}_+^α by

$$-\mathcal{I}_+^\alpha(v) = -\Phi_+(\alpha)v.$$

Then,

$$-\mathcal{I}_+^\alpha v_1 = Iv_1, \quad -\mathcal{I}_+^\alpha v_2 = -Iv_2.$$

Let V_1 be the subspace of V spanned by v_1 and $-Iv_1$ and let V_2 be the subspace of V spanned by v_2 and $-Iv_2$. Then,

$$\begin{aligned} V_1 &= \{v \in V : \mathcal{I}_+^\alpha v = -\mathcal{I}_-^\beta v\}, \quad V_2 = \{v \in V : \mathcal{I}_+^\alpha v = \mathcal{I}_-^\beta v\}, \\ V &= V_1 \oplus V_2. \end{aligned}$$

2.2 Twistor space

Let TV be the tangent bundle of V and let $T_v V$ be the tangent space of V at v . We identify $T_v V$ with V in the usual manner. We denote the integrable hypercomplex structures and the Riemannian metric induced from V by the same symbols: (J_1, J_2, J_3) , $(\tilde{J}_1, \tilde{J}_2, \tilde{J}_3)$ and $\langle \cdot, \cdot \rangle$ respectively. Then, $\langle \cdot, \cdot \rangle$ is Hermitian with respect to (J_1, J_2, J_3) and $(\tilde{J}_1, \tilde{J}_2, \tilde{J}_3)$.

Let $\tilde{A}_0 = (A_0, -J_1 A_0, -J_3 A_0, -J_2 A_0)$ be an orthonormal ordered frame that corresponds to \tilde{v}_0 . Then, $iA_0 = A_0 i$, $jA_0 = A_0 j$ and $kA_0 = A_0 k$. The set of all orthonormal ordered frames (A_1, A_2, A_3, A_4) with the same orientation as \tilde{A}_0 constitute a principal $\text{SO}(4)$ -bundle P over V . The set of all orthonormal ordered frames of the form $(A_1, -IA_1, A_2, -IA_2)$ with orthogonal almost complex structure I of TV and the same orientation as \tilde{A}_0 constitutes a principal $\text{U}(2)$ -bundle Q over V . Then, Q is identified with a section of the fiber bundle

$$\pi^V: Z \rightarrow V, \quad Z = P \times_{\text{SO}(4)} \text{SO}(4)/\text{U}(2) = V \times \text{SO}(4)/\text{U}(2).$$

The bundle Z is referred to as the *twistor space* of V . The set of all sections of π^V is considered to be the set of all almost complex structures of V . For an orthogonal almost complex structure I of V , we obtain the orthonormal ordered frame $(A_1, -IA_1, A_2, -IA_2)$, which corresponds to the map $(a, b): V \rightarrow \text{Sp}(1) \times \text{Sp}(1)$ by the equation

$$\begin{aligned} (A_1, -IA_1, A_2, -IA_2) &= (A_0, -J_1 A_0, -J_3 A_0, -J_2 A_0) \phi(a, b), \\ -I &= -I_+(\alpha) = -I_-(\beta), \quad \alpha = a^\flat, \quad \beta = (b^{-1})^\sharp. \end{aligned}$$

Let \tilde{P} be the spinor structure of P . Then, \tilde{P} is the $\text{Sp}(1) \times \text{Sp}(1)$ -bundle, which is the lift of P by ϕ . Selecting $\phi|_{\text{U}(1) \times \text{Sp}(1)}$ for the double covering of $\text{U}(2)$ and considering V to be the right complex vector space V_+ , the twistor space is identified with the fiber bundle

$$\begin{aligned} \tilde{\pi}_+^V: \tilde{Z}_+ &\rightarrow V, \\ \tilde{Z}_+ &= \tilde{P} \times_{\text{Sp}(1) \times \text{Sp}(1)} (\text{Sp}(1) \times \text{Sp}(1)) / (\text{U}(1) \times \text{Sp}(1)) \\ &= V \times \text{Sp}(1) / \text{U}(1) \cong V \times \mathbb{P}(V_+). \end{aligned}$$

Let $J_{\mathbb{P}(V_+)}$ be the complex structure of $\mathbb{P}(V_+)$. An integrable complex structure $J_{\tilde{Z}_+}$ of \tilde{Z}_+ is defined by

$$\begin{aligned} J_{\tilde{Z}_+}(A, S) &= (-I_+(\alpha)A, J_{\mathbb{P}(V_+)}S), \\ (v, \alpha) &\in V \times \mathbb{P}(V_+), \quad (A, S) \in T_{(v, \alpha)}(V \times P(V_+)) \cong T_v V \times T_\alpha P(V_+). \end{aligned}$$

Selecting $\phi|_{\text{Sp}(1) \times \text{U}(1)}$ for the double covering of $\text{U}(2)$ and considering V to be the left complex vector space V_- , the twistor space is also identified with the fiber bundle

$$\begin{aligned} \tilde{\pi}_-^V: \tilde{Z}_- &\rightarrow V, \\ \tilde{Z}_- &= \tilde{P} \times_{\text{Sp}(1) \times \text{Sp}(1)} (\text{Sp}(1) \times \text{Sp}(1)) / (\text{Sp}(1) \times \text{U}(1)) \\ &= V \times \text{Sp}(1) / \text{U}(1) \cong V \times \mathbb{P}(V_-). \end{aligned}$$

Let $J_{\mathbb{P}(V_-)}$ be the complex structure of $\mathbb{P}(V_-)$. Using a similar discussion, we obtain the complex structure $J_{\tilde{Z}_-}$ of \tilde{Z}_- , which is defined by

$$\begin{aligned} J_{\tilde{Z}_-}(A, S) &= (-I_-(\beta)A, J_{\mathbb{P}(V_-)}S), \\ (v, \beta) &\in V \times \mathbb{P}(V_-), \quad (A, S) \in T_{(v, \beta)}(V \times P(V_-)) \cong T_v V \times T_\beta P(V_-). \end{aligned}$$

3 Conformal maps

We explain conformal maps by orthogonal complex structures and revisit the definition of conformal maps by Pedit and Pinkall [26].

Let Σ be a Riemann surface with the complex structure J_Σ . Recall that a holomorphic function on Σ is a conformal map $h: \Sigma \rightarrow \mathbb{C}$ such that $dh \circ J_\Sigma =$

$i dh = dh i$. We consider a conformal map from Σ to V as an analog of a holomorphic function. For the map $f: \Sigma \rightarrow V$, denote by Υ_f the set of all maps μ from Σ such that $\mu(p)$ is an orthogonal complex structure of $T_{f(p)}V$ for each $p \in \Sigma$.

Definition 1. We refer to a non-constant map $f: \Sigma \rightarrow V$ a *conformal map* if there exists a map $I^\Sigma \in \Upsilon_f$ such that $df \circ J_\Sigma = -I^\Sigma df$.

If a conformal map f is not an immersion at p , then df is the zero map at p and p is a branch point of f .

Let $T\Sigma$ be the tangent bundle of Σ and let $T_p\Sigma$ be the tangent space of Σ at p . Then, the tangent space of f at p is $df(T_p\Sigma)$. Denote the normal space of f at p by $(df(T_p\Sigma))^\perp$. The twistor space of V explains conformal maps as follows.

Theorem 1. *If $f: \Sigma \rightarrow V$ is a conformal map with $df \circ J_\Sigma = -I^\Sigma df$ for a map $I^\Sigma \in \Upsilon_f$, then maps $\alpha: \Sigma \rightarrow \text{Sp}(1)/\text{U}(1)$ and $\beta: \Sigma \rightarrow \text{Sp}(1)/\text{U}(1)$ exist such that*

$$I^\Sigma = I_+(\alpha) = I_-(\beta).$$

At each point p of Σ , the open set U including p , local lifts $a: U \rightarrow \text{Sp}(1)$ and $b: U \rightarrow \text{Sp}(1)$ for α and β , respectively, and a complex $(1,0)$ -form η on U exist such that

$$df = ak\eta b^{-1}.$$

Proof. We only have to prove the theorem for a point in which f is an immersion.

Assume that f is an immersion at $p \in \Sigma$. Because I^Σ is an orthogonal complex structure of V , the maps $\alpha: \Sigma \rightarrow \text{Sp}(1)/\text{U}(1)$ and $\beta: \Sigma \rightarrow \text{Sp}(1)/\text{U}(1)$ exist such that $I^\Sigma = I_+(\alpha) = I_-(\beta)$ per the discussion in Section 2. Because $df(T_p\Sigma)$ is preserved by $I^\Sigma(p)$, we may assume that the existence the open set U , including p , and the ordered orthogonal local frame of TV on U of the form

$$(A_1, -I^\Sigma A_1, A_2, -I^\Sigma A_2)$$

such that $df(T\Sigma)$ is framed by A_2 and $-I^\Sigma A_2$. The maps $a: U \rightarrow \text{Sp}(1)$ and $b: U \rightarrow \text{Sp}(1)$ exist such that

$$(A_1, -I^\Sigma A_1, A_2, -I^\Sigma A_2) = (A_0, -J_1 A_0, -J_3 A_0, -J_2 v_0) \phi(a, b),$$

$$a^\flat = \alpha, \quad (b^{-1})^\sharp = \beta.$$

Because $I^\Sigma = \mathcal{I}_+^\alpha = \mathcal{I}_-^\beta$ on $df(T_q\Sigma)$ for each $q \in U$, we obtain

$$df \circ J_\Sigma = -aia^{-1}df = dfbib^{-1}.$$

Because

$$a^{-1}(df \circ J_\Sigma)b = -ia^{-1}dfb = a^{-1}dfbi,$$

the complex $(1,0)$ -form η exists such that $a^{-1}dfb = k\eta$. Therefore, $df = ak\eta b^{-1}$. \square

Because $\phi(a, a)$ preserves V_c^\perp for each a , we immediately obtain a three-dimensional version of Theorem 1.

Corollary 1. *If $f: \Sigma \rightarrow V_c^\perp$ is a conformal map with $df \circ J_\Sigma = -I^\Sigma df$ for the map $I^\Sigma \in \Upsilon_f$, the maps $\alpha: \Sigma \rightarrow \text{Sp}(1)/\text{U}(1)$ and $\beta: \Sigma \rightarrow \text{Sp}(1)/\text{U}(1)$ exist such that*

$$I^\Sigma = I_+(\alpha) = I_-(\beta).$$

At each point p of Σ , the open set U , including p , a map $a: U \rightarrow \text{Sp}(1)$ with $a^\flat = \alpha$ and $(a^{-1})^\sharp = \beta$ and the complex $(1, 0)$ -form η on U exist such that

$$df = ak\eta a^{-1}.$$

By Theorem 1, we arrive the definition of conformal maps in Pedit and Pinkall [26]:

Lemma 1. *A non-constant map $f: \Sigma \rightarrow V$ is a conformal map if and only if the maps $N, \tilde{N}: \Sigma \rightarrow \text{Im } \mathbb{H} \cap \text{Sp}(1)$ exist such that $df \circ J_\Sigma = N df = -df \tilde{N}$.*

Proof. Assume that $f: \Sigma \rightarrow V$ is a conformal map with $df \circ J_\Sigma = -I^\Sigma df = -I_+(\alpha) df = -I_-(\beta) df$. Let $N = -\Phi_+(\alpha)$ and $\tilde{N} = -\Phi_-(\beta)$. By Theorem 1, we obtain $df \circ J_\Sigma = N df = -df \tilde{N}$.

For the maps $N, \tilde{N}: \Sigma \rightarrow \text{Im } \mathbb{H} \cap \text{Sp}(1)$, we obtain the maps $\alpha = \Phi_+^{-1}(N)$ and $\beta = \Phi_-^{-1}(\tilde{N})$. Let a and b be local maps such that $a^\flat = \alpha$ and $(b^{-1})^\sharp = \beta$. Then $N = -aia^{-1}$ and $\tilde{N} = -bib^{-1}$. In addition, $\phi(a, b)$ defines the map $I^\Sigma \in \Upsilon_f$ such that $df \circ J_\Sigma = -I^\Sigma df$. \square

$$\begin{array}{ccccc}
\text{Sp}(1) & \xrightarrow{b} & \text{Sp}(1)/\text{U}(1) & \xleftarrow{\tilde{\pi}} & \tilde{Z}_+ = V \times \text{Sp}(1)/\text{U}(1) \\
H \downarrow & \swarrow a & \uparrow \alpha & \nearrow \tilde{f}_+ = (f, \alpha) & \downarrow \pi^{\tilde{Z}_+} \\
S^2 & \xleftarrow{N = -\Phi_+(\alpha)} & \Sigma & \xrightarrow{f} & V \\
& & df \circ J_M = -I^\Sigma df = \mathcal{I}_{a^\flat}^\Sigma df = N df. & &
\end{array}$$

Figure 1: conformal map and twistor space

We review a holomorphic map by this formulation. A holomorphic map $f: \Sigma \rightarrow V_+$ is a conformal map with $df \circ J_\Sigma = -J_1 df = df i$ and a holomorphic map $f: \Sigma \rightarrow V_-$ is a conformal map with $df \circ J_\Sigma = -\tilde{J}_1 df = -i df$.

An orthogonal almost complex structure of V is preserved by a conformal transformation of V . Thus to analyze a conformal map by its lift to the twistor space is a natural idea. We distinguish the following lifts:

Definition 2. Let $f: \Sigma \rightarrow V$ be a conformal map with $df \circ J_\Sigma = -I^\Sigma df$ and $\alpha: \Sigma \rightarrow \text{Sp}(1)/\text{U}(1)$ and $\beta: \Sigma \rightarrow \text{Sp}(1)/\text{U}(1)$ be maps with $I^\Sigma = I_+(\alpha) = I_-(\beta)$. We refer to $\tilde{f}_+ = (f, \alpha): \Sigma \rightarrow \tilde{Z}_+$ as a *left canonical lift* of $f: \Sigma \rightarrow V$ and $\tilde{f}_- = (f, \beta): \Sigma \rightarrow \tilde{Z}_-$ as a *right canonical lift* of $f: \Sigma \rightarrow V$.

A left or right canonical lift is referred to a twistor lift in [7] and [11].

Definition 3. Let $f: \Sigma \rightarrow V$ be a conformal map with left canonical lift (f, α) and right canonical lift (f, β) . Assume that $a: \Sigma \rightarrow \text{Sp}(1)$ and $b: \Sigma \rightarrow \text{Sp}(1)$ are maps with $a^\flat = \alpha$ and $(b^{-1})^\sharp = \beta$. We refer to $df = ak\eta b^{-1}$ with the complex $(1, 0)$ -form η on Σ as a *canonical factorization* of df by a , b^{-1} and η .

A canonical factorization is local because a , b^{-1} and η are locally defined.

4 Local conformal maps

We investigate properties of local conformal maps by a canonical factorization.

We assume that Σ is a simply-connected open subset of \mathbb{C} . We denote the standard holomorphic coordinate of \mathbb{C} by z . Then, a $(1, 0)$ -form is $c dz$ for a complex function c . Then, $\eta = -ka^{-1}dfb = c dz$ for a complex function c . Theorem 1 delivers a method of construction for a conformal map.

Lemma 2. *If the maps $a, b: \Sigma \rightarrow \text{Sp}(1)$ and the complex $(1, 0)$ -form η on Σ satisfies*

$$da \wedge k\eta b^{-1} + ak d\eta b^{-1} - ak\eta \wedge db^{-1} = 0, \quad (1)$$

a conformal map $f: \Sigma \rightarrow V$ exists with $df = ak\eta b^{-1}$.

Proof. Differentiating the one-form $ak\eta b^{-1}$, we obtain

$$d(ak\eta b^{-1}) = da \wedge k\eta b^{-1} + ak d\eta b^{-1} - ak\eta \wedge db^{-1}.$$

If the maps $a, b: \Sigma \rightarrow \text{Sp}(1)$ and the complex $(1, 0)$ -form η satisfies the equation (1), then the map $f: \Sigma \rightarrow V$ exists with $df = ak\eta b^{-1}$. Because $df \circ J_\Sigma = -aia^{-1}df = dfbib^{-1}$, the map f is conformal. \square

In the following section, we assume that the conformal map f has a canonical factorization $df = ak\eta b^{-1}$. The maps a , b^{-1} and the one-form η of a canonical factorization $df = ak\eta b^{-1}$ are not uniquely determined. If u and v are maps from Σ to $\text{U}(1)$, then $(au)^\flat = a^\flat$ and $(vb^{-1})^\sharp = (b^{-1})^\sharp$. Then,

$$df = ak\eta b^{-1} = (au)(u^{-1}k\eta v^{-1})(bv^{-1})^{-1} = (au)k(u\eta v^{-1})(bv^{-1})^{-1}.$$

Let $\Omega^{(1,0)}$ be the set of all complex one-forms of type $(1, 0)$ on Σ . Then, $\text{U}(1)$ acts on $\Omega^{(1,0)}$ by multiplication. For a conformal map f with canonical factorization $df = ak\eta b^{-1}$, we obtain the unique triplet $(a^\flat, (b^{-1})^\sharp, [\eta])$, which consisting of $a^\flat, (b^{-1})^\sharp: \Sigma \rightarrow \text{Sp}(1)/\text{U}(1)$ and $[\eta] \in \Omega^{(1,0)}/\text{U}(1)$.

By Lemma 2, we obtain a representation formula for a conformal map $f: \Sigma \rightarrow V$ with the canonical factorization $df = ak \eta b^{-1}$:

$$\begin{aligned} f(p) &= \int_{\gamma} ak \eta b^{-1} + f(p_0), \\ da \wedge k \eta b^{-1} + ak d\eta b^{-1} - ak \eta \wedge db^{-1} &= 0, \\ a, b: \Sigma &\rightarrow \text{Sp}(1), \quad \eta \circ J_{\Sigma} = i \eta = \eta i. \end{aligned}$$

Here, γ is a path from p_0 to p . The zeros of η are the branch points of f .

We fix the canonical factorization $df = ak \eta b^{-1}$. If a $(1, 0)$ -form η is nowhere vanishing, then η is a global section of a real line bundle $l(\eta) = \{r\eta_p : p \in \Sigma, r \in \mathbb{R}\}$ with the projection $\pi_{l(\eta)}: l(\eta) \rightarrow \Sigma$, $\pi_{l(\eta)}(r\eta_p) = p$.

Lemma 3. *Let $f: \Sigma \rightarrow V$ be a conformal immersion with canonical factorization $df = ak \eta b^{-1}$. Let \tilde{f} be an orientation-preserving conformal transform of f in V . Then, the canonical factorization $d\tilde{f} = \tilde{a}k \tilde{\eta} \tilde{b}^{-1}$ exists such that $\tilde{a}^{-1} d\tilde{a} = a^{-1} da$, $\tilde{b}^{-1} d\tilde{b} = b^{-1} db$ and $l(\tilde{\eta}) = l(\eta)$.*

If \tilde{f} is a Euclidean motion of f , then the canonical factorization $d\tilde{f} = \tilde{a}k \tilde{\eta} \tilde{b}^{-1}$ exists with $\tilde{\eta} = \eta$.

Proof. The map

$$\tilde{f} = \lambda f \mu^{-1} + \nu \quad (\lambda, \mu \in \mathbb{H} \setminus \{0\}, \nu \in V)$$

is a conformal transform of f . The differential of \tilde{f} is

$$d\tilde{f} = \lambda df \mu^{-1} = \lambda ak \eta b^{-1} \mu^{-1} = \frac{\lambda}{|\lambda|} ak \frac{|\lambda|}{|\mu|} \eta b^{-1} \frac{\mu^{-1}}{|\mu|^{-1}},$$

Thus the canonical factorization $d\tilde{f} = \tilde{a}k \tilde{\eta} \tilde{b}^{-1}$ with

$$\tilde{a} = \frac{\lambda}{|\lambda|} a, \quad \tilde{b} = \frac{\mu}{|\mu|} b, \quad \tilde{\eta} = \frac{|\lambda|}{|\mu|} \eta$$

satisfies $\tilde{a}^{-1} d\tilde{a} = a^{-1} da$, $\tilde{b}^{-1} d\tilde{b} = b^{-1} db$ and $l(\tilde{\eta}) = l(\eta)$.

A Euclidean motion \tilde{f} of f is

$$\tilde{f} = \lambda f \mu^{-1} + \nu \quad (\lambda, \mu \in \text{Sp}(1), \nu \in V).$$

Then, we obtain the factorization

$$d\tilde{f} = \lambda df \mu^{-1} = (\lambda a)k \eta (\mu b)^{-1}.$$

Because $|\lambda a| = |\mu b| = 1$, this result is a canonical factorization with $\tilde{\eta} = \eta$. \square

Because we can fix the complex $(1, 0)$ -form η under Euclidean motions, the one-form η includes Riemannian geometric information of f . Because the first fundamental form of f is

$$\frac{1}{2}(df \otimes_{\mathbb{R}} d\bar{f} + d\bar{f} \otimes_{\mathbb{R}} df) = \frac{1}{2}(a \eta \otimes_{\mathbb{R}} \bar{\eta} a^{-1} + b \bar{\eta} \otimes_{\mathbb{R}} \eta b^{-1}),$$

the $(1,0)$ -one-form η can generally describe a part of the Riemannian geometric properties. If the codomain of f is included in V_c^\perp , then the $(1,0)$ form completely explains the Riemannian geometric properties.

Lemma 4. *If $df = ak\eta b^{-1}$ is a canonical factorization of the conformal map $f: \Sigma \rightarrow V_c^\perp$, then the first fundamental form is*

$$\frac{1}{2}(\eta \otimes_{\mathbb{R}} \bar{\eta} + \bar{\eta} \otimes_{\mathbb{R}} \eta).$$

Proof. If the codomain of f is contained in V_c^\perp , we may assume that $a = b$. Then, the first fundamental form is

$$\begin{aligned} \frac{1}{2}(a\eta \otimes_{\mathbb{R}} \bar{\eta} a^{-1} + a\bar{\eta} \otimes_{\mathbb{R}} \eta a^{-1}) &= \frac{1}{2}a(\eta \otimes_{\mathbb{R}} \bar{\eta} + \bar{\eta} \otimes_{\mathbb{R}} \eta)a^{-1} \\ &= \frac{1}{2}(\eta \otimes_{\mathbb{R}} \bar{\eta} + \bar{\eta} \otimes_{\mathbb{R}} \eta). \end{aligned}$$

□

If the codomain of f is not contained in V_c^\perp , then η is insufficient for explaining the Riemannian geometric properties of f . However, we observe that the area of f is described by η as follows:

We denote the L^2 -norm of a one-form ω by $\|\omega\|_\Sigma$:

$$\|\omega\|_\Sigma = \left(- \int_\Sigma \omega \wedge (\bar{\omega} \circ J_\Sigma) \right)^{1/2}.$$

In the space of all square integrable one-forms, an inner product is defined as

$$\langle\langle \omega_1, \omega_2 \rangle\rangle_\Sigma = -\frac{1}{2} \int_\Sigma (\omega_1 \wedge \bar{\omega}_2 \circ J_\Sigma + \omega_2 \wedge \bar{\omega}_1 \circ J_\Sigma).$$

For the conformal map $f: \Sigma \rightarrow V$, we denote the area element of f by dA and denote the area of f by $A(f)$. Let $z = x + iy$ be a local holomorphic coordinate of Σ such that (x, y) is a local real coordinate. Then,

$$\begin{aligned} dA &= \sqrt{|f_x|^2 |f_y|^2 - \langle f_x, f_y \rangle} dx \wedge dy = |f_x| |f_y| dx \wedge dy = -\frac{1}{2} df \wedge (d\bar{f} \circ J_\Sigma), \\ 2A(f) &= \|df\|_\Sigma^2. \end{aligned}$$

Lemma 5. *Let $f: \Sigma \rightarrow V$ be a conformal map with the canonical factorization $df = ak\eta b^{-1}$. Then,*

$$2A(f) = \|\eta\|_\Sigma^2.$$

Proof. The area element of f is

$$dA = -\frac{1}{2} ak\eta b^{-1} \wedge (\overline{ak\eta b^{-1}} \circ J_\Sigma) = -\frac{1}{2} \eta \wedge (\bar{\eta} \circ J_\Sigma).$$

Thus, the lemma holds. □

We consider the area of a conformal map from a disk branched at the center. Let $D = \{z \in \mathbb{C} : |z| < 1\}$, and $D_r = \{z \in \mathbb{C} : |z| < r\}$. Let $f = (f_1, f_2, \dots, f_l) : D \rightarrow \mathbb{R}^l$ be a map. Let

$$\frac{\partial f}{\partial z} = \left(\frac{\partial f_1}{\partial z}, \frac{\partial f_2}{\partial z}, \dots, \frac{\partial f_l}{\partial z} \right) : D \rightarrow \mathbb{C}^l.$$

If $(\partial^n f / \partial z^n)(p) = 0$ ($n = 0, \dots, m$) and $(\partial^{m+1} f / \partial z^{m+1})(p) \neq 0$, then the point p is referred to as a zero of f of order m . If $(\partial^n f / \partial z^n)(p) = 0$ ($n = 1, \dots, m$) and $(\partial^{m+1} f / \partial z^{m+1})(p) \neq 0$, then the point p is referred to as a branch point of f of order m .

As a reference case, we consider a holomorphic function. We consider \mathbb{C} to be $kU = \{v_0 k \lambda : \lambda \in \mathbb{C}\}$. A holomorphic function is a map $f : D \rightarrow kU$ with $df \circ J_\Sigma = -i df = df \circ i$. Let $f = kh = k(h_0 + ih_1)$ with real functions h_0 and h_1 . Because $h : D \rightarrow \mathbb{C}$ is a holomorphic function, a point p is a branch point if and only if complex differential

$$\begin{aligned} h_z(p) &= \frac{1}{2} \left(\frac{\partial h_0}{\partial x} - i \frac{\partial h_0}{\partial y} \right) + i \frac{1}{2} \left(\frac{\partial h_1}{\partial x} - i \frac{\partial h_1}{\partial y} \right) \\ &= \frac{1}{2} \left(\frac{\partial h_0}{\partial x} + \frac{\partial h_1}{\partial y} \right) + i \frac{1}{2} \left(-\frac{\partial h_0}{\partial y} + \frac{\partial h_1}{\partial x} \right) = 0. \end{aligned}$$

If p is a branch point of f order m , then p is a zero of h_z of order m . We obtain the following lemma:

Lemma 6. *Let $f = kh : D \rightarrow kU$ be a holomorphic function. Assume that 0 is a branch point of f of order $m - 1$ and $|h_z(z)/z^{m-2}| \leq C$ on D . Then,*

$$A(f|_{D_r}) \leq \pi \frac{C^2}{m} r^{2m} \quad (0 < r < 1).$$

If $z_0 \in D \setminus \{0\}$ exists such that $|h_z(z_0)| = C|z_0|^{m-1}$ or $|(h_z/z^{m-2})_z(0)| = C$, then the equality holds.

Proof. The complex function h_z is a holomorphic function on D with zero of order $m - 1$ at 0 and $|h_z(z)/z^{m-2}| \leq C$. We recall the Schwarz lemma:

Theorem 2 ([28], [12]). *Let $f : D \rightarrow D$ be a holomorphic function such that $f(0) = 0$. Then, $|f(z)| \leq |z|$ on D and $|f_z(0)| \leq 1$. The equality holds if and only if $|f_z(0)| = 1$ or there exists $z_0 \in D \setminus \{0\}$ such that $|f(z_0)| = |z_0|$.*

Applying the Schwarz lemma to $h_z(z)/z^{m-2}$, we obtain $|h_z(z)| \leq C|z|^{m-1}$. The equality holds if and only if $z_0 \in D \setminus \{0\}$ exists such that $|h_z(z_0)| = C|z_0|^{m-1}$ or $|(h_z/z^{m-2})_z(0)| = C$. Then,

$$\begin{aligned} A(f|_{D_r}) &= \|dh\|_{D_r^2} = -\frac{1}{2} \int_{D_r} (dh \wedge (d\bar{h} \circ J_\Sigma)) \\ &= -\frac{1}{2} \int_{D_r} |h_z|^2 dz \wedge (d\bar{z} \circ J_\Sigma) \end{aligned}$$

$$\begin{aligned} &\leq -\frac{1}{2} \int_{D_r} C^2 |z|^{2m-2} dz \wedge (d\bar{z} \circ J_\Sigma) \\ &= \pi \frac{C^2}{m} r^{2m}. \end{aligned}$$

If $z_0 \in D \setminus \{0\}$ exists such that $|h_z(z_0)| = C|z_0|^{m-1}$ or $|(h_z/z^{m-2})_z(0)| = C$, then the equality holds. \square

For the general conformal map $f: D \rightarrow V$ with the canonical factorization $df = ak\eta b^{-1}$, the one-form η is not always a holomorphic one-form. In the case in which f is branched and holomorphic, we obtain the following estimate of the area. If $f: D \rightarrow V_+$ is holomorphic, then the complex functions f_0 and f_1 with $f = f_0 + kf_1$ are holomorphic.

Lemma 7. *Let $f: D \rightarrow V_+$ be a holomorphic map that is branched at 0. Let $f_0: D \rightarrow \mathbb{C}$ and let $f_1: D \rightarrow \mathbb{C}$ be holomorphic functions such that $f = f_0 + kf_1$. Assume that 0 is a branch point of f_0 and f_1 of order $m_0 - 1$ and $m_1 - 1$, respectively. We assume that positive numbers C_0 and C_1 exists such that $|(f_0(z)/z^{m_0-2})_z| \leq C_0$, $|(f_1(z)/z^{m_1-2})_z| \leq C_1$ on D . Then,*

$$A(f|_{D_r}) \leq \pi \left(\frac{C_0^2}{m_0} r^{2m_0} + \frac{C_1^2}{m_1} r^{2m_1} \right) \quad (0 < r < 1).$$

Assume that $z_0 \in D \setminus \{0\}$ exists such that the following equalities hold:

- $|(f_0)_z(z_0)| = C_0|z_0|^{m_0-1}$ or $|((f_0)_z/z^{m_0-2})_z(0)| = C_0$.
- $|(f_1)_z(z_0)| = C_1|z_0|^{m_1-1}$ or $|((f_1)_z/z^{m_1-2})_z(0)| = C_1$.

Then, the equality holds.

Proof. By the Schwarz lemma, we obtain

$$|(f_0)_z(z)| \leq C_0|z|^{m_0-1}, \quad |(f_1)_z(z)| \leq C_1|z|^{m_1-1}$$

on D . The equality holds if and only if $z_0 \in D$ exists such that the following equalities hold:

- $|(f_0)_z(z_0)| = C_0|z_0|^{m_0-1}$ or $|((f_0)_z/z^{m_0-2})_z(0)| = C_0$.
- $|(f_1)_z(z_0)| = C_1|z_0|^{m_1-1}$ or $|((f_1)_z/z^{m_1-2})_z(0)| = C_1$.

By a straightforward calculation, we obtain

$$\begin{aligned} A(f|_{D_r}) &= -\frac{1}{2} \int_{D_r} (df \wedge (d\bar{f} \circ J_\Sigma)) \\ &= -\frac{1}{2} \int_{D_r} (df_0 \wedge (d\bar{f}_0 \circ J_\Sigma) + df_1 \wedge (d\bar{f}_1 \circ J_\Sigma)) \\ &= -\frac{1}{2} \int_{D_r} (|(f_0)_z|^2 + |(f_1)_z|^2) dz \wedge (d\bar{z} \circ J_\Sigma) \end{aligned}$$

$$\begin{aligned}
&\leq -\frac{1}{2} \int_{D_r} (C_0^2 |z|^{2m_0-2} + C_1^2 |z|^{2m_1-2}) dz \wedge (d\bar{z} \circ J_\Sigma) \\
&= \pi \left(\frac{C_0^2}{m_0} r^{2m_0} + \frac{C_1^2}{m_1} r^{2m_1} \right).
\end{aligned}$$

Assume that $z_0 \in D \setminus \{0\}$ exists such that the following equalities holds:

- $|(f_0)_z(z_0)| = C_0 |z_0|^{m_0-1}$ or $|((f_0)_z/z^{m_0-2})_z(0)| = C_0$.
- $|(f_1)_z(z_0)| = C_1 |z_0|^{m_1-1}$ or $|((f_1)_z/z^{m_1-2})_z(0)| = C_1$.

Then, the equality holds. \square

5 Transforms

We assume that Σ is a simply-connected open subset of \mathbb{C} . We recall procedures to construct a conformal map by two given conformal maps.

Assume that $f: \Sigma \rightarrow V$ is a nowhere-vanishing conformal map and $g: \Sigma \rightarrow V$ is a conformal map. Pedit and Pinkall showed in [26] that, if $df \circ J_\Sigma = N df$ and $dg \circ J_\Sigma = N dg$, then the map $h = f^{-1}g$ is a conformal map with $dh \circ J_\Sigma = f^{-1}Nf dh$, and if $df \circ J_\Sigma = -df \tilde{N}$ and $dg \circ J_\Sigma = -dg \tilde{N}$, then the map $h = gf^{-1}$ is a conformal map with $dh \circ J_\Sigma = -dh fNf^{-1}$. This result is used to construct a (Hamiltonian stationary) Lagrangian surface in [22].

In terms of canonical lift, we obtain the following lemma:

Lemma 8. *If the left canonical lift of f is (f, a^b) and the left canonical lift of g is (g, a^b) , then the map $h = f^{-1}g$ is a conformal map with left canonical lift $(h, (|f|f^{-1}a)^b)$. If the right canonical lift of f is $(f, (b^{-1})^\sharp)$ and the right canonical lift of g is $(g, (b^{-1})^\sharp)$, then the map $h = gf^{-1}$ is a conformal map with right canonical lift $(h, (fb/|f|)^{-1})^\sharp$.*

Proof. If the left canonical lift of f is (f, a^b) , then $df \circ J_\Sigma = -aia^{-1}df$. The differential of $h = f^{-1}g$ is

$$dh = f^{-1}(-df f^{-1}g + dg).$$

Thus,

$$dh \circ J_\Sigma = -f^{-1}aia^{-1}f(f^{-1}(-df f^{-1}g + dg)).$$

Therefore, the map $h = f^{-1}g$ is a conformal map with left canonical lift $(h, (|f|f^{-1}a)^b)$.

If the right canonical lift of f is $(f, (b^{-1})^\sharp)$, then $df \circ J_\Sigma = df bib^{-1}$. The differential of $h = gf^{-1}$ is

$$dh = (dg - gf^{-1}df)f^{-1}.$$

Thus,

$$dh \circ J_\Sigma = ((dg - gf^{-1}df)f^{-1})fbib^{-1}f^{-1}.$$

Therefore, the map $h = gf^{-1}$ is a conformal map with right canonical lift $(h, ((fb/|f|)^{-1})^\sharp)$. \square

We cite the following lemma, which is subsequently applied.

Lemma 9 ([5]). *Let ω be a one-form with values in V such that $\omega \circ J_\Sigma = N\omega = -\omega \tilde{N}$ for maps $N, \tilde{N}: \Sigma \rightarrow \text{Im } \mathbb{H} \cap \text{Sp}(1)$.*

If η is a one-form with values in V such that $\eta \circ J_\Sigma = \eta N$, then $\eta \wedge \omega = 0$. If η is a one-form with values in V such that $\eta \circ J_\Sigma = -\tilde{N}\eta$, then $\omega \wedge \eta = 0$.

Assume that ω is nowhere vanishing. If η is a one-form with values in V such that $\eta \wedge \omega = 0$, then $\eta \circ J_\Sigma = \eta N$. If η is a one-form with values in V such that $\omega \wedge \eta = 0$, then $\eta \circ J_\Sigma = -\tilde{N}\eta$.

We translate Lemma 9 into the language of conformal maps and their canonical factorization.

Lemma 10. *Let $f: \Sigma \rightarrow V$ be a conformal map with left canonical lift (f, a^b) and right canonical lift $(f, (b^{-1})^\sharp)$.*

If $h_L: \Sigma \rightarrow V$ is a conformal map with right canonical lift $(h_L, ((ak)^{-1})^\sharp)$, then $dh_L \wedge df = 0$. If $h_R: \Sigma \rightarrow V$ is a conformal map with left canonical lift $(h_R, (bk)^b)$, then $df \wedge dh_R = 0$.

Assume that f is an immersion. If $h_L: \Sigma \rightarrow V$ is a map such that $dh_L \wedge df = 0$, then h_L is a conformal map with right canonical lift $(h_L, ((ak)^{-1})^\sharp)$. If $h_R: \Sigma \rightarrow V$ is a map such that $df \wedge dh_R = 0$, then h_R is a conformal map with left canonical lift $(h_R, (bk)^b)$.

Proof. Because $df \circ J_\Sigma = (-aia^{-1})df = df(bib^{-1})$, Lemma 10 is based on Lemma 9. \square

Definition 4. We refer to the conformal map $h_R: \Sigma \rightarrow V$ with $df \wedge dh_R = 0$ as the *right Bäcklund transform* of f and a conformal map $dh_L: \Sigma \rightarrow V$ with $dh_L \wedge df = 0$ as the *left Bäcklund transform* of f .

In [5], the right Bäcklund transform and the left Bäcklund transform for a Willmore surface are given and called the forward Bäcklund transform and the backward Bäcklund transform respectively.

The Bäcklund transforms for a conformal map of a Riemann surface to S^4 is defined in [20]. Restricting the codomain of a conformal map to S^4 with one point removed and fixing the stereographic projection from the point, the Bäcklund transforms are reduced to Definition 4 (see [24]).

The Darboux transforms of a conformal map of a Riemann surface into S^4 is defined in [3]. In a similar manner, as the Bäcklund transforms, we obtain a Darboux transform of a conformal map of a Riemann surface into \mathbb{E}^4 (see [24]). In terms of canonical lifts, a Darboux transform is explained as follows.

Lemma 11. *Let $f: \Sigma \rightarrow V$ be a conformal map with left canonical lift (f, a^b) and right canonical lift $(f, (b^{-1})^\sharp)$.*

If $h_L: \Sigma \rightarrow V$ is a left Bäcklund transform of f with $dg_L = h_L df$ and nowhere vanishing, then the map $\widehat{f}_L := -h_L^{-1}g_L + f: \Sigma \rightarrow V$ is a conformal map with right canonical lift $(\widehat{f}_L, ((akh_L^{-1}g_L)^{-1}/|h_L^{-1}g_L|)^\#)$.

If $h_R: \Sigma \rightarrow V$ is a right Bäcklund transform of f with $dg_R = df h_R$ and nowhere vanishing, then the map $\widehat{f}_R := -g_R h_R^{-1} + f: \Sigma \rightarrow V$ is a conformal map with left canonical lift $(\widehat{f}_R, (g_R h_R^{-1} b k / |g_R h_R^{-1}|)^\flat)$.

Proof. Because

$$\begin{aligned} d\widehat{f}_L &= -dh_L^{-1}g_L - h_L^{-1}dg_L + df = -dh_L^{-1}g_L, \\ d\widehat{f}_R &= -dg_R h_R^{-1} - g_R dh_R^{-1} + df = -g_R dh_R^{-1}, \end{aligned}$$

the lemma holds. \square

Definition 5. We refer to \widehat{f}_L in Lemma 11 as the *left Darboux transform* of f by a left Bäcklund transform h_L and \widehat{f}_R as the *right Darboux transform* of f by a right Bäcklund transform h_R .

We obtain the following relation between area of a conformal map and the area of its Darboux transform.

Theorem 3. Let $f: \Sigma \rightarrow V$ be a conformal map, let h_L be the right Bäcklund transform of f , let \widehat{f}_L be the right Darboux transform by h_L , let h_R be the right Bäcklund transform of f and let \widehat{f}_R be the right Darboux transform by h_R . Assume that $dg_L = h_L df$ and $dg_R = df h_R$. We assume that f , \widehat{f}_L , df , $d\widehat{f}_L$, $d(h_L^{-1}g_L)$ and $d(g_R h_R^{-1})$ are square integrable. Then

$$\begin{aligned} A(f) + A(\widehat{f}_L) - \langle\langle df, d\widehat{f}_L \rangle\rangle_\Sigma &= \frac{\|d(h_L^{-1}g_L)\|_\Sigma^2}{2}, \\ A(f) + A(\widehat{f}_R) - \langle\langle df, d\widehat{f}_R \rangle\rangle_\Sigma &= \frac{\|d(g_R h_R^{-1})\|_\Sigma^2}{2}. \end{aligned}$$

Proof. By the definition of a left Darboux transform, we obtain

$$d(f - \widehat{f}_L) = df - d\widehat{f}_L = d(h_L^{-1}g_L).$$

Thus,

$$\|df - d\widehat{f}_L\|_\Sigma^2 = \|d(h_L^{-1}g_L)\|_\Sigma^2$$

Then,

$$2A(f) + 2A(\widehat{f}_L) - 2\langle\langle df, d\widehat{f}_L \rangle\rangle_\Sigma = \|d(h_L^{-1}g_L)\|_\Sigma^2.$$

Then, we obtain the former equality. In a similar manner, we have the latter equality. \square

6 The Weierstrass representation

We connect the canonical factorization with the Weierstrass representation by Pedit and Pinkall ([26], Theorem 4.3) and obtain a global representation of a differential of a conformal map.

Let $f: \Sigma \rightarrow V$ be a conformal map with left canonical lift (f, α) and right canonical lift (f, β) . Assume that $df = ak\eta b^{-1}$ is a canonical factorization.

Let L and \tilde{L} be the trivial right quaternionic line bundles over Σ with fiber V . Define a real bilinear pairing $(\ , \) : L \otimes_{\mathbb{R}} \tilde{L} \rightarrow T^*\Sigma \otimes_{\mathbb{R}} V$ by

$$(v_0\lambda, v_0\mu) = \bar{\lambda} df \mu \quad (\lambda, \mu \in \mathbb{H}),$$

where $\otimes_{\mathbb{R}}$ indicates the tensor product over \mathbb{R} . The quaternionic linear complex structures J_L and $J_{\tilde{L}}$ exist for L and \tilde{L} , respectively, such that

$$\begin{aligned} (v_0, v_0) \circ J_{\Sigma} &= df \circ J_{\Sigma} \\ &= -\Phi_+(\alpha) df = df \Phi_-(\beta) \\ &= -\Phi_+(\alpha) (v_0, v_0) = (v_0, v_0) \Phi_-(\beta) \\ &= (v_0 \Phi_+(\alpha), v_0) = (v_0, v_0 \Phi_-(\beta)) \\ &= (J_L v_0, v_0) = (v_0, J_{\tilde{L}} v_0). \end{aligned}$$

Define the quaternionic holomorphic structures D_L and $D_{\tilde{L}}$ for L and \tilde{L} , respectively, by

$$\begin{aligned} D_L(v_0\lambda) &= v_0 \frac{1}{2} (d\lambda + \Phi_+(\alpha) d\lambda \circ J_{\Sigma}), \\ D_{\tilde{L}}(v_0\mu) &= v_0 \frac{1}{2} (d\mu + \Phi_-(\beta) d\mu \circ J_{\Sigma}) \\ &(\lambda, \mu: \Sigma \rightarrow \mathbb{H}). \end{aligned}$$

Then,

$$\begin{aligned} d(v_0\lambda, v_0\mu) &= \frac{1}{2} (d\bar{\lambda} - d\bar{\lambda} \circ J_{\Sigma} \Phi_+(\alpha)) \wedge (v_0, v_0)\mu \\ &\quad - (v_0\lambda, v_0) \wedge \frac{1}{2} (d\mu + \Phi_-(\beta) d\mu \circ J_{\Sigma}) \end{aligned}$$

Then, for any nowhere-vanishing holomorphic section $v_0\lambda$ of L and $v_0\mu$ of \tilde{L} , the pairing $(v_0\lambda, v_0\mu)$ is a closed one-form such that

$$\begin{aligned} (v_0\lambda, v_0\mu) \circ J_{\Sigma} &= -\Phi_+(\tilde{\alpha}) (v_0\lambda, v_0\mu) = (v_0\lambda, v_0\mu) \Phi_-(\tilde{\beta}), \\ \tilde{\alpha} &= \left(\frac{\bar{\lambda}a}{|\lambda|} \right)^{\flat}, \quad \tilde{\beta} = \left(\frac{\mu b^{-1}}{|\mu|} \right)^{\sharp}. \end{aligned}$$

If $(v_0\lambda, v_0\mu)$ is exact, then a conformal map $g: \Sigma \rightarrow V$ exists with canonical lifts $(g, \tilde{\alpha})$ and $(g, \tilde{\beta})$:

$$dg = (v_0\lambda, v_0\mu), \quad dg \circ J_{\Sigma} = -\Phi_+(\tilde{\alpha}) dg = dg \Phi_-(\tilde{\beta}).$$

The branch points of g are the branch points of f .

Let $E_L = \{\psi \in L : J_L \psi = \psi i\}$ and let $E_{\tilde{L}} = \{\psi \in \tilde{L} : J_{\tilde{L}} \psi = \psi i\}$. The bundle E_L and $E_{\tilde{L}}$ are the eigenbundles of J_L and $J_{\tilde{L}}$, respectively.

Lemma 12. *Let $f: \Sigma \rightarrow V$ be a conformal map with the canonical factorization $df = ak \eta b^{-1}$. Then, $v_0 a$ and $v_0 b$ are sections of E_L and $E_{\tilde{L}}$ respectively.*

Proof. Because $df = ak \eta b^{-1}$, we obtain

$$\eta = -ka^{-1} df b = -ka^{-1} (v_0, v_0) b.$$

Then,

$$\begin{aligned} \eta \circ J_\Sigma &= -k(J_L(v_0 a), v_0) b = -ka^{-1}(v_0, J_{\tilde{L}}(v_0 b)) \\ &= -k(-i)(v_0 a, v_0) b = -ka^{-1}(v_0, v_0) i. \end{aligned}$$

Thus, $J_L(v_0 a) = v_0 a i$ and $J_{\tilde{L}}(v_0 b) = v_0 b i$. \square

If the codomain of f is contained in V_c^\perp , then we obtain $L = \tilde{L}$, $J_L = J_{\tilde{L}}$ and $a = b$. The eigenbundle E is a spinor bundle of Σ .

7 Constrained Willmore surfaces

In the following sections, we link a property of a conformal map to a property of its canonical lift.

Let $f: \Sigma \rightarrow V$ be a conformal immersion with left canonical lift (f, α) and right canonical lift (f, β) . Define the maps $N, \tilde{N}: \Sigma \rightarrow \text{Im } \mathbb{H} \cap \text{Sp}(1)$ by $df \circ J_\Sigma = N df = -df \tilde{N}$. Denote the mean curvature vector of f by \mathcal{H} , the Gaussian curvature by K and the area element by dA . The differential df , \mathcal{H} and dA have the following relation:

Lemma 13 ([5], Proposition 8).

$$df \overline{\mathcal{H}} = -\frac{1}{2}(dN \circ J_\Sigma + N dN), \quad \overline{\mathcal{H}} df = \frac{1}{2}(d\tilde{N} \circ J_\Sigma + \tilde{N} d\tilde{N}).$$

Let R^\perp be the normal curvature tensor of f and let K^\perp be the normal curvature defined by

$$K^\perp = \langle R^\perp(X, J_\Sigma X) \xi, N \xi \rangle$$

for a unit tangent vector field X and a unit normal vector field ξ . Then,

$$W(f) = \int_\Sigma (|\mathcal{H}|^2 - K - K^\perp) dA$$

is referred to as the Willmore functional of f . A conformal map f is referred to as a Willmore conformal map if it is a critical conformal map of the Willmore functional for compactly supported infinitesimal variations. Critical points of

W for compactly supported infinitesimal conformal variations are referred to as a conformally constrained Willmore conformal map. We refer to a conformally constrained Willmore conformal map as a constrained Willmore conformal map. In terms of the twistor space and canonical lifts, we obtain the following theorem:

Theorem 4. *Let $f: \Sigma \rightarrow V$ be a conformal immersion with canonical lifts $\tilde{f}_+ = (f, \alpha)$ and $\tilde{f}_- = (f, \beta)$. Define the maps $N, \tilde{N}: \Sigma \rightarrow \text{Im } \mathbb{H} \cap \text{Sp}(1)$ by $df \circ J_\Sigma = N df = -df \tilde{N}$. The following statements are mutually equivalent :*

- (i) *the lift \tilde{f}_+ or \tilde{f}_- is a harmonic section (vertically harmonic),*
- (ii) *the map α or β is a harmonic map,*
- (iii) *the map N or \tilde{N} is a harmonic map, and*
- (iv) *the map f is constrained Willmore.*

Proof. Because the twistor space of V is a product bundle, the equivalence between (i) and (ii) is trivial. Note that N is harmonic if and only if $N * dN$ is a closed one-form. Let ∇^\perp be the normal connection of f . Bohle [2] demonstrated that

$$(\nabla^\perp \mathcal{H}) \circ J_\Sigma = N \nabla^\perp \mathcal{H}$$

if and only if N is harmonic and

$$(\nabla^\perp \mathcal{H}) \circ J_\Sigma = (\nabla^\perp \mathcal{H}) \tilde{N}$$

if and only if \tilde{N} is harmonic.

The former equation indicates that \mathcal{H} is holomorphic with respect to the Koszul-Malgrange holomorphic structure:

$$(\nabla^\perp)^{(0,1)} \mathcal{H} = \frac{1}{2} (\nabla^\perp \mathcal{H} + N (\nabla^\perp \mathcal{H}) \circ J_\Sigma) = 0.$$

The latter equation indicates that \mathcal{H} is holomorphic with respect to the Koszul-Malgrange holomorphic structure:

$$(\nabla^\perp)^{(0,1)} \mathcal{H} = \frac{1}{2} (\nabla^\perp \mathcal{H} + ((\nabla^\perp \mathcal{H}) \circ J_\Sigma) \tilde{N}) = 0.$$

The statement (iii) and (iv) are equivalent to the following statement: \mathcal{H} is holomorphic with respect to $(\nabla^\perp)^{(0,1)}$. The statement (i) is equivalent to the following statement: \mathcal{H} is holomorphic with respect to $(\nabla^\perp)^{(0,1)}$ by Corollary 5.4 in [13]. \square

Hélein and Romon [14] analyzed a representation formula for Lagrangian surfaces. In their formula, twistor lifts are implicitly employed. Leschke and Romon [21] use this formula to study spectral curves of Hamiltonian stationary Lagrangian surfaces. By Theorem 4, we observe that a Hamiltonian stationary Lagrangian surface is constrained Willmore. In [6], Hamiltonian stationary

Lagrangian surfaces are considered in terms of twistor lifts. We modify the formula in [21] and clarify the role of twistor lifts using canonical factorization.

We consider V to be a two-dimensional complex vector space by \tilde{J}_2 and consider the symplectic structure of V . Then, we obtain the following corollary.

Corollary 2. *The conformal map $f: \Sigma \rightarrow V$ with left canonical lift (f, α) is Lagrangian with a Lagrangian angle δ if and only if $\alpha = (e^{(\delta/2)j}k)^\flat$ for the map $\delta: \Sigma \rightarrow \mathbb{R}$. In addition, f is Hamiltonian stationary if and only if δ is harmonic.*

Proof. Leschke and Romon showed that if $f: \Sigma \rightarrow V$ is a Lagrangian conformal immersion, then $df = e^{(\delta/2)j} dz e^u q$ for the local holomorphic coordinate $z = x + iy$ of Σ , the map $q: \Sigma \rightarrow \text{Sp}(1)$ and the map $u: \Sigma \rightarrow \mathbb{R}$ such that

$$df \otimes df = e^u(dx \otimes dx + dy \otimes dy).$$

In addition, f is Hamiltonian stationary if and only if δ is harmonic.

We observe that

$$df = e^{(\delta/2)j} dz e^u q = e^{(\delta/2)j} k k (-dz) e^u q,$$

Exchange $-dz e^u$ with a complex-valued $(1, 0)$ -form η and relabel q with b^{-1} . Then, we obtain the canonical factorization $df = e^{(\delta/2)j} k k \eta b^{-1}$ for a Lagrangian conformal map. Thus, the left canonical lift of f is (f, α) with $\alpha = (e^{(\delta/2)j}k)^\flat$. \square

8 Super-conformal map

We discuss constrained Willmore conformal maps with holomorphic canonical lift. Prior to our discussion of conformal maps, we investigate the map $N = -aia^{-1}: \Sigma \rightarrow S^2 = \text{Im } \mathbb{H} \cap \text{Sp}(1)$ with $a: \Sigma \rightarrow \text{Sp}(1)$.

We consider $S^2 = \text{Im } \mathbb{H} \cap \text{Sp}(1)$ to be the Riemann sphere \mathbb{CP}^1 . Let w the stereographic projection from the north pole. Then

$$w \mapsto \frac{2 \text{Re } w}{|w|^2 + 1} i + \frac{2 \text{Im } w}{|w|^2 + 1} j + \frac{|w|^2 - 1}{|w|^2 + 1} k$$

is a holomorphic parametrization of $S^2 \setminus \{k\}$.

The following lemma is proven in [25]. We provide an alternate short proof.

Lemma 14. *The map $N: \Sigma \rightarrow \text{Im } \mathbb{H} \cap \text{Sp}(1) \cong \mathbb{CP}^1$ is holomorphic if and only if $dN \circ J_\Sigma = -N dN = dN N$.*

The map $N: \Sigma \rightarrow \text{Im } \mathbb{H} \cap \text{Sp}(1) \cong \mathbb{CP}^1$ is anti-holomorphic if and only if $dN \circ J_\Sigma = N dN = -dN N$.

Proof. Let (x, y) be a local conformal coordinate of Σ with

$$J_\Sigma \frac{\partial}{\partial x} = \frac{\partial}{\partial y}.$$

The map $N: \Sigma \rightarrow \text{Im } \mathbb{H} \cap \text{Sp}(1)$ is holomorphic if and only if the vector product $N \times N_x$ is equal to $-N_y$. Differentiating $N^2 = -1$, we obtain $N dN = -(dN)N$. Then, N is holomorphic if and only if $dN \circ J_\Sigma = -N dN = (dN)N$. Similarly, N is anti-holomorphic if and only if $dN \circ J_\Sigma = N dN = -(dN)N$. \square

This lemma is translated as follows:

Lemma 15. *Let $\alpha: \Sigma \rightarrow \text{Sp}(1)/\text{U}(1)$ be a map and let $a: \Sigma \rightarrow \text{Sp}(1)$ be a map with $a^\flat = \alpha$. Let $N = -\Phi_+(\alpha)$. The map $N: \Sigma \rightarrow \text{Im } \mathbb{H} \cap \text{Sp}(1) \cong \mathbb{C}P^1$ is holomorphic if and only if the map $\alpha: \Sigma \rightarrow \text{Sp}(1)/\text{U}(1) \cong \mathbb{P}(V_+)$ is anti-holomorphic. The map $N: \Sigma \rightarrow \text{Im } \mathbb{H} \cap \text{Sp}(1) \cong \mathbb{C}P^1$ is anti-holomorphic if and only if the map $\alpha: \Sigma \rightarrow \text{Sp}(1)/\text{U}(1) \cong \mathbb{P}(V_+)$ is holomorphic.*

Let $\beta: \Sigma \rightarrow \text{Sp}(1)/\text{U}(1)$ be a map and $b: \Sigma \rightarrow \text{Sp}(1)$ be a map with $(b^{-1})^\sharp = \beta$. Let $\tilde{N} = -\Phi_-(\beta)$. A map $\tilde{N}: \Sigma \rightarrow \text{Im } \mathbb{H} \cap \text{Sp}(1) \cong \mathbb{C}P^1$ is holomorphic if and only if $\beta: \Sigma \rightarrow \text{Sp}(1)/\text{U}(1) \cong \mathbb{P}(V_-)$ is anti-holomorphic. A map $\tilde{N}: \Sigma \rightarrow \text{Im } \mathbb{H} \cap \text{Sp}(1) \cong \mathbb{C}P^1$ is anti-holomorphic if and only if $\beta: \Sigma \rightarrow \text{Sp}(1)/\text{U}(1) \cong \mathbb{P}(V_-)$ is holomorphic.

A variant of this lemma is also proven in [25]. We provide an improved proof.

Proof. The map a^\flat is holomorphic if and only if the map $c: \Sigma \rightarrow \mathbb{C} \setminus \{0\}$ exists such that the map $av_0c: \Sigma \rightarrow \mathbb{P}(V_+)$ is holomorphic. Thus, a^\flat is holomorphic if and only if the map $c: \Sigma \rightarrow \mathbb{C} \setminus \{0\}$ exists such that the map $ac: \Sigma \rightarrow V_+$ is holomorphic. The map $ac: \Sigma \rightarrow V_+$ is holomorphic if and only if $d(ac) \circ J_\Sigma = d(ac)i$. The differential of $N = -aia$ is

$$dN = d(-(ac)i(ac)^{-1}) = (ac)(-(ac)^{-1}d(ac)i + i(ac)^{-1}d(ac))(ac)^{-1}.$$

Thus, if ac is holomorphic, then $dN \circ J_\Sigma = -dN N = N dN$. If a^\flat is anti-holomorphic, then N is anti-holomorphic. If N is anti-holomorphic, then $dN \circ J_\Sigma = N dN = -dN N$. Thus,

$$\begin{aligned} & (-(ac)^{-1}d(ac)i + i(ac)^{-1}d(ac)) \circ J_\Sigma \\ &= (-(ac)^{-1}d(ac)i + i(ac)^{-1}d(ac))i \\ &= -i(-(ac)^{-1}d(ac)i + i(ac)^{-1}d(ac)). \end{aligned}$$

Therefore, we can select c such that ac is holomorphic. Then, a^\flat is holomorphic.

Similarly, a^\flat is anti-holomorphic if and only if N is holomorphic.

The map $(b^{-1})^\sharp$ is holomorphic if and only if the map $c: \Sigma \rightarrow \mathbb{C} \setminus \{0\}$ exists such that a map $c^{-1}b^{-1} = (bc)^{-1}: \Sigma \rightarrow V_-$ is holomorphic. The map $(bc)^{-1}: \Sigma \rightarrow V_-$ is holomorphic if and only if $d(bc)^{-1} = -i d(bc)^{-1}$. The differential of $\tilde{N} = -bib^{-1}$ is

$$d\tilde{N} = d(-(bc)i(bc)^{-1}) = (bc)(d(bc)^{-1}(bc)i - i d(bc)^{-1}(bc))(bc)^{-1}.$$

Thus, if $(bc)^{-1}$ is holomorphic, then $d\tilde{N} \circ J_\Sigma = -d\tilde{N} \tilde{N} = \tilde{N} d\tilde{N}$. Therefore, if $(b^{-1})^\sharp$ is holomorphic, then \tilde{N} is anti-holomorphic. If \tilde{N} is anti-holomorphic, then $d\tilde{N} \circ J_\Sigma = \tilde{N} d\tilde{N} = -d\tilde{N} \tilde{N}$. Thus,

$$\begin{aligned} & (d(bc)^{-1}(bc)i - i d(bc)^{-1}(bc)) \circ J_\Sigma \\ &= (d(bc)^{-1}(bc)i - i d(bc)^{-1}(bc))i \\ &= -i(d(bc)^{-1}(bc)i - i d(bc)^{-1}(bc)). \end{aligned}$$

Therefore, we can select c such that $(bc)^{-1}$ is holomorphic. Then, $(b^{-1})^\sharp$ is holomorphic. \square

A conformal map is referred to as a *super-conformal* map if its curvature ellipse is a circle at each immersed point. As shown in [23], a super-conformal map is a Bäcklund transform of a minimal surface. A super-conformal map is referred to as an associated Willmore surface in [19]. A holomorphic function is a super-conformal map. Let $f: \Sigma \rightarrow V$ be a conformal map with $df \circ J_\Sigma = N df = -df \tilde{N}$. The curvatures of f are calculated by N and \tilde{N} in [5]. We observe that the following lemma holds from [5], Section 8.2.

Lemma 16. *A conformal map f is super-conformal if and only if N or \tilde{N} is anti-holomorphic.*

If f is super-conformal, then a holomorphic lift of f to the twistor space exists (see, for example, [5], Theorem 5). We have distinguished this holomorphic lift.

Theorem 5. *The left canonical lift or the right canonical lift of a conformal map is holomorphic if and only if the conformal map is super-conformal.*

Proof. If f is conformal, then f is always holomorphic with respect to I^Σ . If the left canonical lift (f, α) is holomorphic, then α is holomorphic. If (f, α) is holomorphic, then f is super-conformal. Similarly, if the right canonical lift (f, β) is holomorphic, then f is super-conformal.

Conversely, if f is super-conformal with $df \circ J_\Sigma = N df = -df \tilde{N}$, then N or \tilde{N} is anti-holomorphic by Lemma 16. If $N = -\Phi_+(\alpha)$, then α is holomorphic by Lemma 15. Similarly, if $\tilde{N} = -\Phi_-(\beta)$, then β is holomorphic by Lemma 15. \square

Let $f: \Sigma \rightarrow V$ be a super-conformal map with canonical factorization $df = ak\eta b^{-1}$. We may assume that $a: \Sigma \rightarrow V_+$ is holomorphic. Then, the local complex function c exists such that $d(ac) \circ J_\Sigma = d(ac)i$. We obtain the factorization $df = ack\zeta$ with $\zeta = \bar{c}^{-1}\eta b^{-1}$. Differentiating $df = ack\zeta$, we obtain

$$0 = d(df) = d(ac)k \wedge \zeta + ack d\zeta.$$

The branch points of f is exactly the zeros of ac or the zeros of ζ . We employ this factorization for an estimate of the area.

Theorem 6. Let $f: D \rightarrow V$ be a super-conformal map that is branched at 0. Assume that f has the factorization $df = \tilde{a} \zeta$ by a one-form ζ and a holomorphic map $\tilde{a}: D \rightarrow V_+$. Let a_0 and a_1 be holomorphic functions such that $\tilde{a} = a_0 + ka_1$. Assume that 0 is a zero of a_0 and a_1 of order $m_0 - 1$ and $m_1 - 1$ respectively. Assume that positive numbers C_{a_0} , C_{a_1} and C_ζ exist such that $|a_0(z)/z^{m_0-2}| \leq C_{a_0}$, $|a_1(z)/z^{m_1-2}| \leq C_{a_1}$ and $\zeta \wedge (\bar{\zeta} \circ J_\Sigma) \geq C_\zeta dz \wedge (d\bar{z} \circ J_\Sigma)$ on D . Then

$$A(f|_{D_r}) \leq \pi C_\zeta \left(\frac{C_{a_0}^2}{m_0} r^{2m_0} + \frac{C_{a_1}^2}{m_1} r^{2m_1} \right) \quad (0 < r < 1).$$

Assume that $\zeta \wedge (\bar{\zeta} \circ J_\Sigma) = C_\zeta dz \wedge (d\bar{z} \circ J_\Sigma)$ and $z_0 \in D \setminus \{0\}$ exists such that the following equalities hold:

- $|a_0(z_0)| = C_{a_0} |z_0|^{m_0-1}$ or $|((a_0)_z/z^{m_0-2})_z(0)| = C_{a_0}$.
- $|a_1(z_0)| = C_{a_1} |z_0|^{m_1-1}$ or $|((a_1)_z/z^{m_1-2})_z(0)| = C_{a_1}$.

Then, the equality holds.

Proof. By the Schwarz lemma, we obtain $|a_0(z)| \leq C_{a_0} |z|^{m_0-1}$ and $|a_1(z)| \leq C_{a_1} |z|^{m_1-1}$. The equality simultaneously holds if and only if $z_0 \in D \setminus \{0\}$ exists such that the following equalities hold:

- $|a_0(z_0)| = C_{a_0} |z_0|^{m_0-1}$ or $|((a_0)_z/z^{m_0-2})_z(0)| = C_{a_0}$.
- $|a_1(z_0)| = C_{a_1} |z_0|^{m_1-1}$ or $|((a_1)_z/z^{m_1-2})_z(0)| = C_{a_1}$.

By the Schwarz inequality, the area of $f|_{D_r}$ is

$$\begin{aligned} A(f|_{D_r}) &= -\frac{1}{2} \int_{D_r} df \wedge (d\bar{f} \circ J_\Sigma) = -\frac{1}{2} \int_{D_r} |\tilde{a}|^2 \zeta \wedge (\bar{\zeta} \circ J_\Sigma) \\ &\leq -\frac{1}{2} C_\zeta \int_{D_r} (|a_0|^2 + |a_1|^2) dz \wedge (d\bar{z} \circ J_\Sigma) \\ &\leq -\frac{1}{2} C_\zeta \int_{D_r} (C_{a_0}^2 |z|^{2m_0-2} + C_{a_1}^2 |z|^{2m_1-2}) dz \wedge (d\bar{z} \circ J_\Sigma) \\ &= \pi C_\zeta \left(\frac{C_{a_0}^2}{m_0} r^{2m_0} + \frac{C_{a_1}^2}{m_1} r^{2m_1} \right). \end{aligned}$$

Because a is holomorphic, the condition for the equality is based on the condition for the equality in the Schwarz lemma. \square

9 Minimal surfaces

In the previous section, we discussed the case in which α or β is holomorphic. In this section, we discuss minimal surfaces, where α and β are anti-holomorphic.

Theorem 7. Let $f: \Sigma \rightarrow V$ be a conformal map with left canonical lift (f, α) and right canonical lift (f, β) . Define the maps $N, \tilde{N}: \Sigma \rightarrow \text{Im } \mathbb{H} \cap \text{Sp}(1)$ by $df \circ J_\Sigma = N df = -d\tilde{f} \tilde{N}$. The following statements are mutually equivalent :

- (i) the map f is minimal,
- (ii) the map $\alpha: \Sigma \rightarrow \mathrm{Sp}(1)/\mathrm{U}(1) \cong \mathbb{P}(V_+)$ is anti-holomorphic,
- (iii) the map $\beta: \Sigma \rightarrow \mathrm{Sp}(1)/\mathrm{U}(1) \cong \mathbb{P}(V_-)$ is anti-holomorphic,
- (iv) the map N is holomorphic,
- (v) the map \tilde{N} is holomorphic, and
- (vi) at each point of Σ , the local lift a and the local lift b^{-1} of α and β , respectively, local complex nowhere-vanishing functions c, \tilde{c} and a holomorphic one-form ξ exist such that $ac: \Sigma \rightarrow V_+$ and $(b\tilde{c})^{-1}: \Sigma \rightarrow V_-$ are anti-holomorphic and $df = \text{ack } \xi (b\tilde{c})^{-1}$.

Note that the equivalence among (i), (ii) and (iii) is obtained in [8].

Proof. The equivalence among (i), (ii), (iii), (iv) and (v) is trivial by Lemma 13, Lemma 14 and Lemma 15.

The maps α and β are anti-holomorphic if and only if, at each point of Σ , the local $\mathrm{Sp}(1)$ -valued maps a and B with $a^b = \alpha$ and $(b^{-1})^\sharp = \beta$ and the local complex nowhere-vanishing functions c, \tilde{c} exist such that ac and $(b\tilde{c})^{-1}$ are anti-holomorphic:

$$d(ac) \circ J_\Sigma = d(ac) (-i), \quad d(b\tilde{c})^{-1} \circ J_\Sigma = i d(b\tilde{c})^{-1}.$$

These equations are equivalent to

$$\begin{aligned} (d(ac) (ac)^{-1}) \circ J_\Sigma &= (d(ac) (ac)^{-1}) (-aia^{-1}), \\ (b\tilde{c}) d(b\tilde{c})^{-1} \circ J_\Sigma &= (bib^{-1}) (b\tilde{c}) d(b\tilde{c})^{-1}. \end{aligned} \tag{2}$$

The one-form $\xi = -k(ac)^{-1} df (\tilde{c}b)$ is holomorphic if and only if

$$\begin{aligned} (-k(ac)^{-1} df (b\tilde{c})) \circ J_\Sigma &= i(-k(ac)^{-1} df (b\tilde{c})) = (-k(ac)^{-1} df (b\tilde{c}))i, \\ d(-k(ac)^{-1} df (b\tilde{c})) &= k(ac)^{-1} d(ac) (ac)^{-1} \wedge df (b\tilde{c}) \\ &\quad + k(ac) df \wedge (b\tilde{c}) d(b\tilde{c})^{-1} (b\tilde{c}) = 0. \end{aligned} \tag{3}$$

The equations (2) are equivalent to (3) by Lemma 10 and the assumption for f . Thus, statement (vi) is equivalent to statement (i). \square

In Theorem 7, we obtained a local factorization of a differential of a minimal conformal map $df = \tilde{a}k\xi\tilde{b}^{-1}$ by the anti-holomorphic maps $\tilde{a}: \Sigma \rightarrow V_+$ and $\tilde{b}^{-1}: \Sigma \rightarrow V_-$ and a complex holomorphic one-form ξ . A branch point of f is a zero of \tilde{a}, ξ or \tilde{b}^{-1} .

Definition 6. We refer to the factorization $df = \tilde{a}k\xi\tilde{b}^{-1}$ in Theorem 7 as a *factorization* of df .

The factorization of the differential of a minimal conformal map is variant of representations of the differential of a conformal map in [17] and [29].

We modify the factorization around a branch point.

Lemma 17. *Assume that Σ is a simply-connected open subset of \mathbb{C} and let $f: \Sigma \rightarrow V$ be a minimal conformal map. Let p be the only branch point of f . Then, the factorization $df = \mathbf{a}k\tilde{\xi}\mathbf{b}^{-1}$ exists such that \mathbf{a} and \mathbf{b}^{-1} do not vanish around p and $\tilde{\xi}$ is a holomorphic one-form vanishing at p .*

Proof. Let z be a holomorphic coordinate of Σ with $z(p) = 0$ and $df = \tilde{a}k\xi\tilde{b}^{-1}$. Then, anti-holomorphic maps $\mathbf{a}: \Sigma \rightarrow V_+$ and $\mathbf{b}^{-1}: \Sigma \rightarrow V_-$ exist such that they do not vanish at p , $\tilde{a} = \mathbf{a}\bar{z}^m$ and $\tilde{b}^{-1} = z^n\mathbf{b}^{-1}$ with nonnegative integers m and n . Then,

$$df = \mathbf{a}\bar{z}^m k \xi z^n \mathbf{b}^{-1} = \mathbf{a}k \xi z^{m+n} \mathbf{b}^{-1}.$$

Replacing ξz^{n+m} with $\tilde{\xi}$, we obtain a factorization $df = \mathbf{a}k\tilde{\xi}\mathbf{b}^{-1}$ such that \mathbf{a} and \mathbf{b}^{-1} do not vanish at p and $\tilde{\xi}$ is a holomorphic one-form vanishing at p . \square

If the codomain of f is V_c^\perp , we obtain

$$\overline{df} = \overline{\tilde{a}k\xi\tilde{b}^{-1}} = -\overline{\tilde{b}^{-1}}\overline{\tilde{\xi}}\overline{k\tilde{a}} = -\overline{\tilde{b}^{-1}}k\xi\tilde{a} = -\tilde{a}k\xi\tilde{b}^{-1}.$$

Thus, $\overline{\tilde{b}^{-1}}k\tilde{a} = \tilde{a}k\tilde{b}^{-1}$.

We employ this factorization of a differential of a minimal conformal map for an estimate of its area. The branch points of f is exactly the zeros of h . Recall a lower bound of the area of a minimal surface by Alexander and Osserman:

Theorem 8 ([1]). *If $f: D \rightarrow \mathbb{R}^n$ is a minimal conformal map that satisfies*

- $f(0) = c$,
- f has a branch point of order $m - 1$ ($m \geq 1$) at $z = 0$,
- $R > 0$ exists such that $\liminf_{n \rightarrow \infty} |f(z_n)| \geq R$ whenever $|z_n| \rightarrow 1$.

Then,

$$A(f) \geq m\pi(R^2 - |c|^2).$$

Equality holds if and only if the image lies on a plane through the point c , is orthogonal to the line through 0 and c , and, in terms of a complex coordinate w in this plane with c at the origin, is of the form $w = \sqrt{R^2 - |c|^2}z^m$.

We slightly modify this theorem as follows:

Lemma 18. *If $f: D \rightarrow \mathbb{R}^n$ is a minimal conformal map that satisfies*

- $f(0) = c$,
- f has a branch point of order $m - 1$ ($m \geq 1$) at $z = 0$,
- $R_r > 0$ exists such that $\liminf_{n \rightarrow \infty} |f(z_n)| \geq R_r$ whenever $|z_n| \rightarrow r$.

Then,

$$A(f|_{D_r}) \geq m\pi(R_r^2 - |c|^2).$$

Equality holds if and only if the image lies on a plane through the point c , is orthogonal to the line through 0 and c , and, in terms of a complex coordinate w in this plane with c at the origin, is of the form $w = \sqrt{R_r^2 - |c|^2} z^m$.

Proof. Let $g: D \rightarrow \mathbb{R}^n$ be the conjugate minimal surface of f . Then, $\Phi: D \rightarrow \mathbb{C}^n$, $\Phi = (\Phi_1, \Phi_2, \dots, \Phi_n)$, $\Phi_l = f_l + ig_l$ ($l = 1, 2, \dots, n$) is a holomorphic map. Alexander and Osserman [1] demonstrated that

$$A(f|_{D_s}) \geq \frac{\pi}{2} \sum_{k=m+1}^{\infty} (k-m) \sum_{l=1}^n |a_{lk}|^2 s^{2k} - m\pi|c|^2 + m\pi \min_{|w|=s} \sum_{l=1}^n (f_l(w))^2,$$

$$c = (c_1, c_2, \dots, c_n), \quad \Phi_l = c_l + \sum_{k=m}^{\infty} a_{nk} w^k.$$

Then, we obtain

$$A(f|_{D_r}) = \liminf_{s \rightarrow r} A(f|_{D_s}) \geq \frac{\pi}{2} \sum_{k=m+1}^{\infty} (k-m) \sum_{l=1}^n |a_{lk}|^2 r^{2k} + m\pi(R_r^2 - |c|^2) \geq m\pi(R_r^2 - |c|^2).$$

The proof for the condition for the equality is similar to [1]. \square

We provide an upper bound of the area of a minimal conformal map as follows:

Theorem 9. *Let $f: D \rightarrow V$ be a minimal conformal map. Assume that f has the factorization $df = \mathbf{a}k h dz \mathbf{b}^{-1}$ by anti-holomorphic maps $\mathbf{a}: D \rightarrow V_+$ and $\mathbf{b}^{-1}: D \rightarrow V_-$ and the complex holomorphic function h on D . Assume that*

- $f(0) = c$,
- f has the only branch point of order $m-1$ ($m \geq 1$) at $z = 0$,
- $0 < |\mathbf{a}| \leq C_{\mathbf{a}}$, $0 < |\mathbf{b}^{-1}| \leq C_{\mathbf{b}^{-1}}$ and $|h(z)/z^{m-2}| \leq C_h$,
- $R_r > 0$ exists such that $\liminf_{n \rightarrow \infty} |f(z_n)| \geq R_r$ whenever $|z_n| \rightarrow r$ for each r ($0 < r < 1$).

Then,

$$m\pi(R_r^2 - |c|^2) \leq A(f|_{D_r}) \leq \pi \frac{C_{\mathbf{a}}^2 C_{\mathbf{b}^{-1}}^2 C_h^2}{m} r^{2m} \quad (0 < r < 1).$$

Assume that $|\mathbf{a}| = C_{\mathbf{a}}$, $|\mathbf{b}^{-1}| = C_{\mathbf{b}^{-1}}$ and there exists $z_0 \in D \setminus \{0\}$ such that $|h(z_0)| = C_h |z_0|^{m-1}$ or $|(h/z^{m-2})_z(0)| = C_h$. Then, the equality in the right inequality holds.

Proof. By Lemma 18, we obtain

$$m\pi(R_r^2 - |c|^2) \leq A(f|_{D_r})$$

Because 0 is the only branch point of f of order $m - 1$, the point 0 is the zero of h of order $m - 1$. By the Schwarz lemma, we have $|h(z)| \leq C_h |z|^{m-1}$. The equality holds if and only if $z_0 \in D \setminus \{0\}$ exists such that $|h(z_0)| = C_h$ or $|(h/z^{m-2})_z(0)| = C_h$.

The area element of f is

$$-\frac{1}{2} df \wedge (d\bar{f} \circ J_\Sigma) = -\frac{1}{2} |\mathfrak{a}|^2 |\mathfrak{b}^{-1}|^2 |h|^2 dz \wedge (d\bar{z} \circ J_\Sigma)$$

The area of $f|_{D_r}$ is

$$\begin{aligned} A(f|_{D_r}) &\leq \int_{D_r} -\frac{1}{2} C_{\mathfrak{a}}^2 C_{\mathfrak{b}^{-1}}^2 C_h^2 |z|^{2m-2} dz \wedge (d\bar{z} \circ J_\Sigma) \\ &= \pi \frac{C_{\mathfrak{a}}^2 C_{\mathfrak{b}^{-1}}^2 C_h^2}{m} r^{2m}. \end{aligned}$$

If $|\mathfrak{a}| = C_{\mathfrak{a}}$, $|\mathfrak{b}^{-1}| = C_{\mathfrak{b}^{-1}}$ and there exists $z_0 \in D \setminus \{0\}$ such that $|h(z_0)| = C_h$ or $|(h/z^{m-2})_z(0)| = C_h$, then the equality holds. \square

If $\mathfrak{a} = \mathfrak{b}^{-1} = 1$, then f is a holomorphic function, which is a planar minimal conformal map. We note that $f_z(z) = kh$ and $|h(z)/z^{m-2}| \leq C_h$. Thus, if a minimal conformal map is planar, then the upper bound for the area of a minimal conformal map is reduced to the upper bound for the area of a holomorphic function.

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